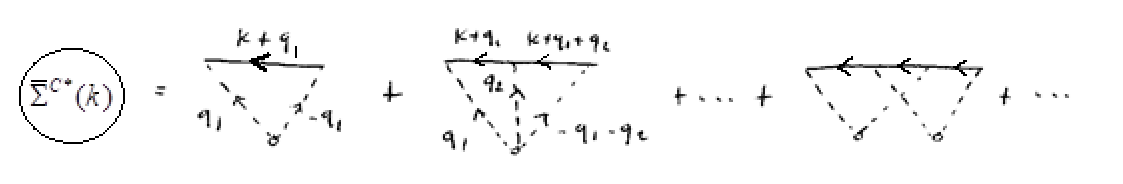
**Weak Disorder**

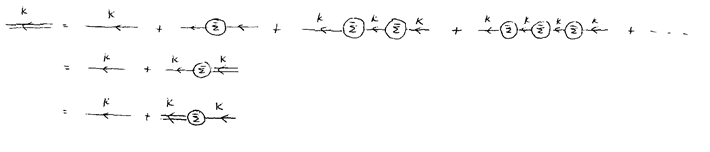
We’ll start by examining the effects the impurities have on the single particle electron states – traveling waves, and on their energies. The way to analyze this is via the self-energy, which will tell us the energy shift and lifetime of these states. This perturbative analysis can only be expected to be valid for weak disorder, i.e., 1/kFℓ << 1. Keep in mind that we should expect the energy shift to be something akin to what we get from Fumi’s theorem (see Quantum Mechanics/Many Identical Particles).

**Self-Energy**

[the proceeding analysis is kind of neglecting the negative signs what would technically be attached to the G’s and V’s in the diagrams, but this is okay, as we avered in the Interaction File, as long as we are consistent] The formidable task of summing all the diagrams entering the expansion of C\*(k,iωn) can be greatly simplified. Define the electron self-energy (strictly the irreducible , or proper self energy), C\*(k,iωn) to be the sum of all irreducible diagrams entering C\*(k,iωn) with their external electron lines removed. Note that momentum k will enter and leave the diagram from left to right, and momentum is flowing against the electron arrows, but along the indicated directions on the impurity lines.



It is simple to see that



The last line contains two alternative formulations of Dyson’s Equation:



Dyson’s Equation can be inverted to give



and so finally,



To obtain the quasi-particle energy and lifetime, we’ll analytically continue to the retarded GF: iωn → ω + i0+. Then,



and look for the pole. We’ll say the pole is at some ω = k. Then we can do a Laurent series expansion about it to say,



and so,



To put a physical interpretation to our result, we can take the inverse (temporal) Fourier transform and obtain:



Zk (the residue at the pole) is the quasi-particle wave function renormalization factor. We can see that Re is the new quasi-particle energy of the state |k>. I would expect that this could be obtained via conventional perturbation theory (after we disorder average the potential and use totally AS many-body wavefunctions). And -2Imk is 1/τ, where τ is the lifetime of the state. This is also apparent when we go ahead and calculate (k,ω).



Note how this will go to the delta function (weighted) as the inverse lifetime goes to 0. We may expect that to first order in the potential, the new pole is just the old pole, and so the simplest approximation to the self-energy is just to evaluate it ‘on shell’, that is at ω = ξk, the unperturbed root.



Then,



Thus Re(k,ξk) gives the energy shift of the quasi-particle, and -2Im(k,ξk) gives the quasi-particle inverse lifetime, at least to first order.

**Σ in the low impurity density limit (T-matrix Approximation)**

Note that ni has dimensions of 1/V. A dimensionless measure of the impurity concentration is ni/kF3 ~ nia03. Now diagrams with q impurities are proportional to niq. So when ni << kF3 we expect diagrams with q >> 1 to be negligible. So we can approximate the self-energy by keeping all diagrams with just one impurity circle/star.



Turns out we can relate this Σ approximation to the T-matrix from elementary Quantum Mechanics. But do not equate Σ with T in general, as that would be neglecting the disorder averaging, Fermi statistics (the electrons are not independent single particles in at least the Pauli Exclusion principle sense), and correlations between single particle scattering events. To facilitate seeing how we relate Σ to T, imagine that we have momentum k flowing in and out. Then the self-energy is:



which is:



(changing variables in the summation q1 → q1 – k, q2 → q2 – q1, q3 → q3 – q2, etc. Now let’s compare to the ‘off-shell’ T-matrix, dotted between two |k> = exp(ikr)/√V states (implicit summation over repeated |k1>’s, |k2>’s). And so note we’ll have that <q|Vi|q´> = Vi(q-q´)/V, where Vi(q-q´) is the Fourier transform of the impurity potential w/r to q-q´.



Now for a spherically symmetric potential, we ought to have Vi(q-q´) = Vi(q´-q). So we can flip the arguments of Vi around.



and then we can see:



and thus,



(remember kk on T refers to expectation between to |k> = eikr/√V states) We can do this another way. We’ll write a recursive equation for the self-energy. If we relax momentum conservation at the impurity vertex to allow for k momentum coming in and k´ coming out, and don’t initially subsume the Vi(0) term into the chemical potential as in the first file, we can write:



and then we could say:



Now compare this to the off-shell T-matrix.



And let’s box it between a <k| and |k´> state. This amounts to the recursive equation:



So then we’d infer that the self energy is,



But then we’d go back and correct our omission of keeping the first order Vi(k - k) term by subtracting it back off, and write:



(again remember kk on T refers to expectation between to |k> = eikr/√V states) But I think it’s customary to switch to kk states whereby |k> = eikr. This factors the volume out of our expression, and then we’ll have:



Remember Vi(k-k) = i(q=0) and has also been referred to as Vi(0) for short and is the Fourier transform of the impurity potential Vi(r) w/r to k-k = 0. When we analytically continue to iωn → ω + i0+ we get the regular off-shell T-matrix from Time Dependent QM perturbation theory (see that constant perturbation file), just multiplied by the impurity density ni. This is what we’d have expected if we had naively neglected all correlations between electrons, i.e., treated them as independent particles, which is reasonable if scattering is relatively unlikely, as it is in the low impurity density limit. We can write this another way. Recall in the QM/Scattering/TDPT file, we wrote down a relationship between the off-shell T matrix and the on-shell T-matrix (the k´ sum we can take to run over all free states of the total H, which should be in 1-1 correspondance with the free states of the unperturbed Hamiltonian, H0).



(where |k>’s are box normalized states = eikr/√V ) Let’s switch to |k> = eikr states instead, to match the convention in our highlighted equation above. This will factor the volume out of all terms (leaving one extra on the sum because of |Tk´k|2) and we’ll have:



And so we can say,



Pretty cool that we have the exact self-energy, in the low density limit at least. And we can use all the QM stuff to work out what the on-shell T-matrix is. From here, we can analytically continue to say, using the identity 1/(x+i0+) = P(1/x) – πiδ(x):



If we evaluate Σ on shell, then we’d set ω = ξk and we find that (P means principle value):



and the real/imaginary parts give us the approximate energy shift and (twice) decay rate. We will of course recognize the decay rate, which we derived in the Quantum Mechanics/Time Dependent/Constant Perturbation file. And might look at the Quantum Mechanics / Scattering / Time Dependent file to recall that we found for this scattering rate (well here below, the kk refers to expectations between block normalized |kblock> = eikr/√V):



Converting the kk expection on T´ to |k> = eikr states, like we have been using above, we’ll have:



Now multiplying through by ni/V we have:



And so we can write the on-shell Σ as,



And so we have:



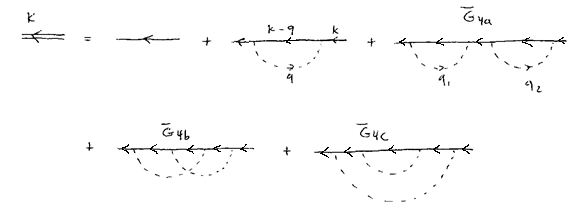
(note we can expect σ to depend on k too, in principal, though in high energy limit it should go to constant ~ 2πR2 I think, for hard sphere potentials?) So we’ll observe that a bunch of electrons in a disorder averaged potential just reduces to *an* electron in a disorder averaged potential in the dilute impurity case. This is because statistics don’t play as much of a role, when there isn’t that much scattering going on, because the electrons have little chance of running into each other.

**Σ in the weak potential (possibly high density) limit (Born Approximation)**

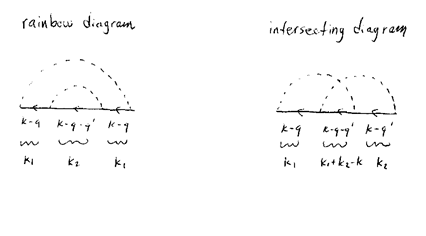
Suppose there is a weak potential, but lots of them (this could be considered a case of high disorder?) In the limit where the first order energy shift is much less than the Fermi energy (the initial unperturbed energy as far as transport is concerned).



We‘ll consider only the lowest order diagrams which include V, because we assume roughly that V, V(r), V(q), is a small parameter and so we include diagrams with only the smallest possible # of V’s – just like we included diagrams above with only the smallest # of impurities. We expect only the lowest – that is the second – order correction to C\*(k,iωn) to be important for electrons close to the Fermi surface, i.e. |k| ≈ kF, usually the case of greatest interest (since Fermi distribution function changes appreciably with temperature only in this region) . In this Born limit we can adopt a simplified diagrammatic notation since each impurity is involved only twice.



Now each dashed line carries a factor of ni|Vi(q)|2. C\*4a is an example of a reducible diagram that can be split in two by cutting a single internal fermionic line. It describes double scattering from one impurity, followed by double scattering from another. C\*4c is an example of a rainbow diagram – an irreducible diagram that has no intersecting lines. If the potential is short ranged this diagram should be smaller than C\*4a since it will require an additional trip between the impurities and the amplitude for such trips is much smaller on average than that for the electron to remain localized in the vicinity of a single impurity. C\*4b is an example of an intersecting diagram – an irreducible diagram that is not a rainbow diagram. If the potential is short ranged, this should be even smaller than C\*4csince it will require yet another trip between the impurities. Note that when impurity lines cross each other in the Feynman diagrams that this signifies interference between scattering between these two impurities. Let’s examine the rainbow and intersecting diagrams below.



As before (see examples next file), only the imaginary parts of these diagrams will give significant contributions - the real parts will be independent of k more or less and can be subsumed into the chemical potential like we did before. We can say this since these integrals will have the same structure as before? *We’re interested in the Green’s function for momenta around the Fermi level so we want k ≈ kF*. And make the following observation. Since:



GC\*0(k,iωn) will be largest for k ~ kF since there the μ will cancel the εk. Therefore GC\*0(k,iωn) will be damped on either side of k = kF. So GC\*0itself discourages deviation from kF. Looking at the rainbow diagram, we see that the integrals over q, q’ will be approximately odd and strongly damped (acting like an effective cutoff for small at small q) so this will restrict k1, k2, and k1 + k2 = k, to near the Fermi surface. Thus…

The rainbow diagram requires that



And the intersecting diagram requires that



The last equality requires consequently that

 since we have set .

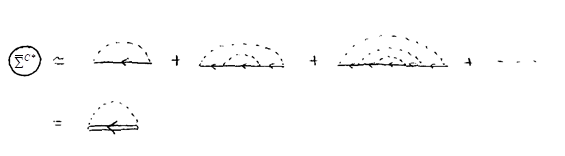
Therefore the k1 and k2 range over practically the entire Fermi sphere for the rainbow diagram, while for the intersecting diagram, k1 and range over the Fermi sphere, but k2 is restricted to a small neighborhood about -k1. We could put this in terms of the q’s too. The rainbow diagram requires that



And the intersecting diagram requires that



The last equality requires consequently that q ≈ qʹ. Thus we expect the rainbow diagram contribution to be larger than the intersecting diagram contribution. More detailed analysis shows that that the intersecting diagram is smaller by a factor of about ~ 1/EFτ ~ 1/kFvFτ = 1/kFℓ. Thus for weak scatterers, and k close to kF, we expect that the rainbow diagrams will be more important than the intersecting ones (but perhaps such crossed diagrams will become important, just as they do for transport, for long lengths?). This suggests an approximation for the self energy in the weak scattering limit. I don’t think this necessarily requires a short ranged potential. Just these two requirements seem to suffice.



where the full propagator in the diagram is evaluated self consistently according to our self energy approximation. Note again that we are summing contributions from an arbitrary number of impurities. This self-consistent approximation can be written explicitly as



i.e.,



where all we have done is to insert the approximation of the self energy into Dyson’s equation.